Geometric Means and Hadamard Products

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Abstract. Ando [1] proved that for m commuting positive definite matrices, the m-fold Hadamard product of their geometric mean is bounded above by their Hadamard product. We obtain a natural extension to the non-commutative case.

In what follows, if M is a positive definite $n \times n$ matrix and $\alpha \in \mathbb{R}$, then M^α will denote its unique positive α^{th} *power*. In 1979, Ando [1] developed a robust definition of the geometric mean of two positive definite $n \times n$ matrices M_1 and M_2 that had been introduced by Pusz and Woronowicz [5]:

$$G(M_1, M_2) := M_2^{1/2} \left(M_2^{-1/2} M_1 M_2^{-1/2} \right)^{1/2} M_2^{1/2}.$$

The geometric mean is symmetric in $M_1,\ M_2$, monotone in each variable and satisfies the arithmetic-geometric mean inequality

$$G(M_1, M_2) \leq A(M_1, M_2) := \frac{1}{2}(M_1 + M_2).$$

Given $0 < \alpha < 1$, the geometric mean is generalized to the α-mean $G^{(\alpha)}(M_1, M_2)$:

$$G^{(\alpha)}(M_1, M_2) := M_2^{1/2}(M_2^{-1/2} M_1 M_2^{-1/2})^\alpha M_2^{1/2}.$$

The usual geometric mean is just $G^{(1/2)}(M_1, M_2)$. In stead of the arithmetic-geometric mean inequality the Young inequality holds for α-mean:

$$G^{(\alpha)}(M_1, M_2) \leq A^{(\alpha)}(M_1, M_2) := \alpha M_1 + (1 - \alpha) M_2.$$

Ando [1] also researched the interaction of this geometric mean with the Hadamard (or Schur) product. If $M = (m_{ij}),\ N = (n_{ij})$ are matrices of the same size, their *Hadamard product* $M \circ N$ is the matrix of entry-wise products:

$$M \circ N := (m_{ij} n_{ij}).$$

Ando [1, Theorem 13] proved that for positive definite $n \times n$ matrices $M,\ N$ we have

$$G(M, N) \circ G(M, N) \leq M \circ N, \tag{A_1}$$

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which, in the commutative case, reduces to

$$(MN)^{\frac{1}{2}} \circ (MN)^{\frac{1}{2}} \leq M \circ N.$$

Using a different method, he succeeded in generalizing his inequality to the case of several commuting positive definite $n \times n$ matrices [1, Theorem 12]:

$$\prod_{1}^{m} \circ \left(\prod_{i=1}^{m} M_i \right)^{\frac{1}{m}} \leq \prod_{i=1}^{m} \circ M_i. \tag{A_2}$$

However, at the time, the notion of geometric mean for several non-commuting matrices was not available; therefore, inequality (A_1) was not developed beyond the case of two matrices.

In 1994, M. Sagae and K. Tanabe [6] successfully developed an approach to the geometric mean — in fact, the weighted geometric mean — of several positive definite matrices. The main purpose of this paper is to blend the ideas of Ando and the concepts of Sagae and Tanabe to extend inequality (A_1) to the case of several positive definite $n \times n$ matrices. As a result, we are able to provide a simpler proof of inequality (A_2).

The definition $G_w(M_1, M_2, \ldots, M_m)$ of the geometric mean of an m-tuple of positive definite matrices $(M_1, M_2, \ldots, M_m)$ $(m > 2)$ was given by Sagae and Tanabe in 1994.

When $m = 2$, define

$$G_w(M_1, M_2) := G^{(w_1)}(M_1, M_2).$$

Suppose that the definition for the case $m - 1$ has been well established. Now given an m-tuple of positive numbers $(w_1, w_2, \ldots, w_m)$ summing to 1, define

$$G_w(M_1, M_2, \ldots, M_m) := G^{\left(\sum_{j=1}^{m-1} w_j\right)} \left(G_{\tilde{w}}(M_1, M_2, \ldots, M_{m-1}), M_m\right)$$

where

$$\tilde{w} = \left(w_1 / \sum_{j=1}^{m-1} w_j, w_2 / \sum_{j=1}^{m-1} w_j, \ldots, w_{m-1} / \sum_{j=1}^{m-1} w_j \right).$$

In order to work effectively with the general geometric mean of Sagae and Tanabe, it is convenient to introduce the $(\alpha_1, \ldots, \alpha_k)$-*power mean* for $(k+1)$-tuple positive definite matrices. Suppose that M_i $(i \geq 1)$ are positive definite $n \times n$ matrices and α_i $(i \geq 1)$ are real scalars. Starting with the two matrix basis, we can continue recursively to define

$$G^{(\alpha_1, \ldots, \alpha_k)}(M_1, \ldots, M_{k+1}) = G^{(\alpha_k)}(G^{(\alpha_1, \ldots, \alpha_{k-1})}(M_1, \ldots, M_k), M_{k+1})$$

for $k \geq 2$.

When $\alpha_i = 1 - \left(w_{i+1} \ / \ \sum_{j=1}^{i+1} w_j \right)$ for $i = 1, \ldots, m-1$, we have

$$G^{(\alpha_1, \ldots, \alpha_{m-1})}(M_1, \ldots, M_m) = G_w(M_1, \ldots, M_m).$$

This general definition of geometric mean has many good properties, but in the case of equal weights it is not symmetric for $k > 2$ (see [3]). For us, it will be significant that the weighted geometric mean satisfies an arithmetic-geometric mean inequality.

Theorem ([6], Theorem 1) *Let w be an m-tuple of positive numbers $(w_1, w_2, \ldots, w_m)$ summing to 1 and let M_i $(1 \le i \le m)$ be positive definite $n \times n$ matrices. Then*

$$G_w(M_1, \ldots, M_k) \le A_w(M_1, \ldots, M_k) := w_1 M_1 + \cdots + w_k M_k.$$

The inequality is strict unless $M_1 = \cdots = M_k$.

From inequality (A_1) it is natural to conjecture that when M_i $(i = 1, 2, 3)$ are positive definite matrices (possibly non-commuting), the inequality

$$G_w(M_1, M_2, M_3) \circ G_w(M_1, M_2, M_3) \circ G_w(M_1, M_2, M_3) \le M_1 \circ M_2 \circ M_3$$

holds true. The answer is positive, and in fact the order of the matrices is unimportant. This is to some extent surprising, since Ando's proof of inequality (A_1) depends on the symmetry of the geometric mean of two matrices. The main result in this paper is:

Theorem 1 *Let w be an m-tuple of positive numbers $(w_1, w_2, \ldots, w_m)$ summing to 1 and let M_i $(1 \le i \le m)$ be positive definite $n \times n$ matrices. If $(i_1, i_2, \ldots, i_m)$, $(j_1, j_2, \ldots, j_m), \ldots, (k_1, k_2, \ldots, k_m)$ are arbitrary m-permutations of $\{1, 2, \ldots, m\}$, then*

$$G_w(M_{i_1}, \ldots, M_{i_m}) \circ G_w(M_{j_1}, \ldots, M_{j_m}) \circ \cdots \circ G_w(M_{k_1}, \ldots, M_{k_m}) \le \prod_{i=1}^{m} \circ M_i.$$

In order to obtain the proof of Theorem 1, we need to develop some properties of tensor products of matrices. If $M = (m_{ij})$ is an $k \times l$ matrix and $N = (n_{ij})$ is an $s \times t$ matrix, then their *tensor (*or *Kronecker) product* is the $ks \times lt$ matrix

$$M \otimes N := \begin{bmatrix} m_{11}N & \cdots & m_{1l}N \\ \vdots & \ldots & \vdots \\ m_{k1}N & \cdots & m_{kl}N \end{bmatrix}.$$

The tensor product of finitely many matrices can be defined by induction.

The basic properties of the tensor product can be found in [2, p. 15] and [1, p. 224]. We need two more properties that we were unable to find in the literature.

Proposition 1 *(i) Let M_i $(1 \le i \le k)$ be $m \times m$ matrices and let N_i $(1 \le i \le k)$ be $n \times n$ matrices. Then*

$$\prod_{i=1}^{k} (M_i \otimes N_i) = \left(\prod_{i=1}^{k} M_i \right) \otimes \left(\prod_{i=1}^{k} N_i \right).$$

(ii) Let M_i be positive definite $n_i \times n_i$ matrices $(1 \le i \le k)$. Then, for any real number α,

$$\left(\prod_{i=1}^{k} \otimes M_i \right)^{\alpha} = \prod_{i=1}^{k} \otimes M_i^{\alpha}.$$

Proof (i) This follows easily by induction from the observation that

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD),$$

which can be verified immediately from the definition of the tensor product of matrices.

(ii) Again, this follows by induction from the case of the product of two matrices. Since M_1 and M_2 are positive definite, we can write (see [4, Theorem 2.5.4]) $M_1 = U_1^* S U_1$ and $M_2 = U_2^* T U_2$, using appropriate unitary matrices U_1 and U_2, and positive definite diagonal matrices S and T. By (i),

$$M_1 \otimes M_2 = (U_1^* \otimes U_2^*)(S \otimes T)(U_1 \otimes U_2) = (U_1 \otimes U_2)^*(S \otimes T)(U_1 \otimes U_2)$$

and, since $U_1 \otimes U_2$ is unitary, the functional calculus (see [1, p. 212]) allows us to write

$$\begin{aligned} & (M_1 \otimes M_2)^\alpha = (U_1 \otimes U_2)^*(S \otimes T)^\alpha (U_1 \otimes U_2) \\ = \;& (U_1^* \otimes U_2^*)(S^\alpha \otimes T^\alpha)(U_1 \otimes U_2) = (U_1^* S U_1)^\alpha \otimes (U_2^* T U_2)^\alpha = M_1^\alpha \otimes M_2^\alpha. \end{aligned}$$

□

In [1], Ando pointed out a fundamental commutativity relation between the geometric mean and tensor product of two positive definite matrices, namely

$$G(M_1 \otimes M_2, N_1 \otimes N_2) = G(M_1, N_1) \otimes G(M_2, N_2). \tag{1}$$

In fact, this can readily be extended to the *α-power mean.* The analog of identity (1) is

$$G^{(\alpha)}(M_1 \otimes M_2, N_1 \otimes N_2) = G^{(\alpha)}(M_1, N_1) \otimes G^{(\alpha)}(M_2, N_2), \tag{2}$$

and this follows easily from Proposition 1.

A multi-stage induction argument leads to a simple, but powerful extension of identity (2):

Proposition 2 *Let M_{ij} $(1 \le i \le m,\ 1 \le j \le k)$ be positive definite $n \times n$ matrices, and let w be an m-tuple of positive numbers $(w_1, w_2, \ldots, w_m)$ summing to 1 . Then*

$$G_w\left(\prod_{j=1}^{k} \otimes M_{1j}, \ldots, \prod_{j=1}^{k} \otimes M_{mj}\right) = \prod_{j=1}^{k} \otimes\, G_w(M_{1j}, \ldots, M_{mj}).$$

Proof With the help of Proposition 1, a preliminary induction shows that

$$G^{(\alpha_1,\ldots,\alpha_{m-1})}(M_1 \otimes N_1, \ldots, M_m \otimes N_m) = G^{(\alpha_1,\ldots,\alpha_{m-1})}(M_1, \ldots, M_m) \otimes G^{(\alpha_1,\ldots,\alpha_{m-1})}(N_1, \ldots, N_m),$$

and then a second induction gives

$$G^{(\alpha_1,\ldots,\alpha_{m-1})}\left(\prod_{j=1}^{k} \otimes M_{1j}, \ldots, \prod_{j=1}^{k} \otimes M_{mj}\right) = \prod_{j=1}^{k} \otimes G^{(\alpha_1,\ldots,\alpha_{m-1})}(M_{1j}, \ldots, M_{mj}).$$

Setting $\alpha_i = 1 - \left(w_{i+1} \;/\; \sum_{j=1}^{i+1} w_j \right)$ for $i = 1, \ldots, m-1$ yields the statement of Proposition 2. We omit the simple details. $\square$

In many situations, properties of tensor products transfer to Hadamard products. This is thanks to an important connection between the two products (e.g. see [1, Lemma 4]): there is a positive linear map Φ_k from n^k-dimensional space to n-dimensional space of matrices such that, for all $n \times n$ matrices M_i $(1 \le i \le k)$,

$$\Phi_k \left(\prod_{i=1}^{k} \otimes M_i \right) = \prod_{i=1}^{k} \circ M_i. \tag{3}$$

With this, we can quickly identify the key to our proof of Theorem 1:

Proposition 3 *Let M_{ij} $(1 \le i \le m,\ 1 \le j \le k)$ be positive definite $n \times n$ matrices, and let w be an m-tuple of positive numbers $(w_1, w_2, \ldots, w_m)$ summing to 1 . Then*

$$\prod_{j=1}^{k} \otimes\, G_w(M_{1j}, \ldots, M_{mj}) \le \sum_{i=1}^{m} w_i \prod_{j=1}^{k} \otimes M_{ij}$$

and

$$\prod_{j=1}^{k} \circ\, G_w(M_{1j}, \ldots, M_{mj}) \le \sum_{i=1}^{m} w_i \prod_{j=1}^{k} \circ M_{ij}.$$

Proof By Sagae and Tanabe's arithmetic-geometric mean inequality and Proposition 2,

$$\prod_{j=1}^{k} \otimes\, G_w(M_{1j}, \ldots, M_{mj}) = G_w \left(\prod_{j=1}^{k} \otimes\, M_{1j}, \ldots, \prod_{j=1}^{k} \otimes\, M_{mj} \right) \le \sum_{i=1}^{m} w_i \prod_{j=1}^{k} \otimes\, M_{ij}.$$

The Hadamard product inequality follows by (3) from the tensor product inequality. $\square$

Proof of Theorem 1 In Proposition 3, let

$$\begin{aligned}
(M_{11}, \ldots, M_{m1}) &= (M_{i_1}, \ldots, M_{i_m}), \\
(M_{12}, \ldots, M_{m2}) &= (M_{j_1}, \ldots, M_{j_m}), \\
&\vdots \\
(M_{1m}, M_{2m}, \ldots, M_{mm}) &= (M_{k_1}, \ldots, M_{k_m}).
\end{aligned}$$

Then, since the Hadamard product is commutative,

$$\begin{aligned}
& G_w(M_{i_1}, \ldots, M_{i_m}) \circ \cdots \circ G_w(M_{k_1}, \ldots, M_{k_m}) \\
= \; & \prod_{j=1}^{m} \circ\, G_w(M_{1j}, \ldots, M_{mj}) \le \sum_{i=1}^{m} w_i \prod_{j=1}^{m} \circ M_{ij} = \sum_{i=1}^{m} w_i \prod_{j=1}^{m} \circ M_j = \prod_{i=1}^{m} \circ M_i.
\end{aligned}$$

$\square$

It is clear that inequality (A_2) is an immediate corollary of Theorem 1. Let $(i_1, i_2, \ldots, i_m), (j_1, j_2, \ldots, j_m), \ldots, (k_1, k_2, \ldots, k_m)$ be arbitrary m-permutations of $\{1, 2, \ldots, m\}$. If M_i $(1 \le i \le m)$ commute and $w_1 = w_2 = \cdots = w_m = 1/m$, then

$$\begin{aligned} G_w(M_{i_1}, \ldots, M_{i_m}) &= G_w(M_{j_1}, \ldots, M_{j_m}) \\ &= \cdots \\ &= G_w(M_{k_1}, \ldots, M_{k_m}) = \left(\prod_{i=1}^{m} M_i \right)^{\frac{1}{m}}. \end{aligned}$$

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